INJ INTERNATIONAL NEUROUROLOGY JOURNAL

An inverse fractional abstract Cauchy problem with nonlocal conditions and with respect to functions

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Abstract

In a Hilbert space , an inverse fractional abstract Cauchy problem with nonlocal conditions and with respect to functions is investigated. Closed operators with strongly continuous semigroups in a Hilbert space are considered. Also, existence and unique solutions of the considered integral equations are studied.

Key words: Abstract fractional integral equations with respect to functions, Nonlocal initial conditions, Inverse Cauchy problem, Partial differential equations.

Introduction

Many authors have investigated the detection of an unidentified state independent source factor in the heassst equation and various parabolic equations [2], [6], [20]-[24]. We need to study an inverse fractional abstract Cauchy problem with nonlocal conditions and with respect to functions in order to solve problems with the theory of viscoelasticity, (see [1], [7], [19], [22]). This article takes aim to investigate an inverse fractional abstract Cauchy problem with nonlocal conditions and with respect to functions in a real Hilbert space H in the shape:

$$f(t) = f(0) + \frac{1}{\Gamma(\gamma)}\int_0^t [\psi(t) \quad \psi(s))^{\gamma-1} Af(s)ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^t [\psi(t) \quad \psi(s))^{\gamma-1} g(s)u(s)ds, \tag{1}$$

with respect to the nonlocal initial condition:

$$f(0) = f_0 + \sum_{i=1}^{k} c_i f(t_i), \tag{2}$$

and the overdetermination condition:

$$(f(t), w) = p(t), \tag{3}$$

where the gamma function is $\Gamma(\cdot)$, $0 < \gamma \leq 1$, ψ is a real bijective function, $\psi(t) \geq 0$, The linear closed operator A is defined on a dense set S in a real Hilbert space H and with values in H, the inner product in H is $(\cdot,\cdot)$, a really unknown function is g, the real function p is known, in H, f_0, w are known elements, an abstract function named u with values in H is given, $0 \leq t_1 < ... < t_k \leq T$, $c_1,...,c_k$ are real numbers. It is assumed that A produces a strongly continuous semigroup $W(t)$ such that $\| \ W(t)\| \leq K$, for all $t \in \ J = [0,T]$, $W(t)$, $v \in \ S$, $\| \ AW(t)v\| \leq \frac{K}{t}\| \ v\|$, for every element v in H, $t \in (0,T]$, and $\| \ \cdot\|$ is the norm in H, (see [3]).
We take into consideration the integral of operator valued functions

$$\Psi(t)=\int_0^\infty \zeta_\gamma(\theta)W(\psi^\gamma(t)\theta)d\theta,$$

and

$$\Psi^*=\gamma\int_0^\infty \theta\psi^{\gamma-1}(t)\zeta_\gamma(\theta)W(\psi^\gamma(t)\theta)d\theta\theta>0,$$

where the probability density function ζ_γ is defined on the interval $(0,\infty)$, (see [3], [4]).

In the set of all linear bounded operators $B(H)$ defined on H, Ψ denotes a uniformly continuous function of t.

We suppose these conditions:

B_1: $f_0, w \in S$ and $u(t)\in S$ $\forall$ $t\in J$,

B_2: $|u_1(t)|\geq u_0$, $t\in J$, if a positive constant is u_0 and $u_1(t)=(u(t),w)$,

B_3: With respect to the norm in H, u and Au are abstract functions that are continuous on J,

B_4: $p(t)\in C(J)$, where the set of all continuous functions on J is denoted by $C(J)$.

In section **2**, we study the problem (1) with the nonlocal condition (2) under the overdetermination condition (3). There are three functions provided: f_0, w and p. In a suitable space, the unknown functions f and g are identified. There were several uses for the nonlocal Cauchy problem and fractional calculus theories [2], [8]-[11], [14]-[18], [23].

Solution representation

According to the overdetermination condition (3), a pair of functions $\{f,g\}$ is a strict solution to the problem (1) with the nonlocal condition (2) if $f\in S$ for every $t\in(0,T]$, $g\in C(J)$ and there is satisfaction for the relations (1)-(3).

Thus, we state that under the overdetermination condition (3), the problem (1) with the nonlocal condition (2) is solvable.

Let the following equation:

$$\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}g(s)ds=v(t)-v(0)+\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}(Pg)(s)ds,$$

(4)

where

$$v(t)=\frac{1}{u_1(t)}p(t),$$

and

$$v(0)=\frac{1}{u_1(t)}p(0),$$

where $p(0)=(f(0),v)$, is the comptability condition and a linear operator named P defined on $C(J)$ with values

$$(Pg)(t)=\frac{-1}{u_1(t)}(Af(t),w). \qquad (5)$$

We want to prove the equivelence between the problem (1)-(3) and (4).

Theorem 2.1 *If all conditions* (B_1-B_4) *are achieved, then the following assumptions are true: (1) Equation (4) has a solution* $g\in C(J)$ *if the problem (1)-(3) can be solved.*

(2) The problem (1)-(3) can be solved if equation (4) has a solution $g \in C(J)$.

Proof. Suppose that the problem (1)-(3) can be solved. By scalarly multiplying both sides of equation (1) by w in H. Multiplying both sides of equation (1) by v scalarly in H, we get

$$(f(t),w)=(f(0),w)+\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}(Af(s),w)ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}g(s)(u(s),w)ds$$

$$=p(0)+\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}(Af(s),w)ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}g(s)u_1(s)ds, \tag{6}$$

from equation (5) and equation (6), we have

$$\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}g(s)ds=\frac{1}{u_1(t)}p(t) \quad \frac{1}{u_1(t)}p(0)$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}(Pg)(s)ds,$$

this proves that equation (4) can be solved by g. By assuming that there is a solution $g \in C(J)$ to equation (4), statement (2) can be proven

When this function is used in equation (1), the resultant problem (1), (2) may be considered as a direct nonlocal problem with an unique solution. The solution is [1], [6]

$$f(t)=\Psi(t)\varphi f_0+\Psi(t)\sum_{i=1}^{k}c_i\int_0^{t_i}\Psi^*(\psi(t_i)-\psi(s))g(s)u(s)ds$$

$$+\int_0^t\Psi^*(\psi(t)-\psi(s))g(s)u(s)ds. \tag{7}$$

Now, we prove that the overdetermination condition (3) is satisfied for f. f and g are known, thus equation (4) represents the following identity

$$\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}g(s)u_1(s)ds=p(t)-p(0)$$

$$-\frac{1}{\Gamma(\gamma)}\int_0^t[\psi(t)-\psi(s)]^{\gamma-1}(Af(s),w)ds, \tag{8}$$

subtracting equation (6) from equation (8), we obtain

$$p(t)=(f(t),w),$$

then f achieves the overdetermination condition (3) and the strict solution to the problem (1)-(3) is the pair $\{f,g\}$. The theorem's proof is now complete.

Theorem 2.2 *There exists a unique strictly solution to the problem (1)-(3) if the conditions* $(B_1 - B_4)$ *are satisfied.*

Proof. By applying the equations (4), (5) and (7), (Formally) we get next the Volterra integral equation

$$\frac{1}{\Gamma(\gamma)}\int_0^t [\psi(t)-\psi(s)]^{\gamma-1} g(s)ds = v(t)-v(0)$$

$$-\frac{1}{\Gamma(\gamma)}\int_0^t [\psi(t)-\psi(s)]^{\gamma-1}\frac{1}{u_1(s)}(\Psi(t)\varphi f_0, A^* w)ds$$

(9) $$-\frac{1}{\Gamma(\gamma)}\int_0^t K(\psi(t)-\psi(s))\frac{1}{u_1(s)}g(s)ds,$$

where

$$K(\psi(t)\quad \psi(s)) = \sum_{i=1}^{k}(\Psi(t)c_i\Psi^*(\psi(t_i)-\psi(s))u(s), A^* w)+(\Psi^*(\psi(t)-\psi(s))u(s), A^* w),$$

where the adjoint of the operator A is A^*. Applying the same technique [1], [3]-[6], [12], [13], [16], We observe that the kernel $K(\psi(t)-\psi(s))$, the functions $u_1^{-1}(t)$ and $v(t)$ are continuous functions of t, s in J. Thus, g on j is the unique continuous solution to the integral Volterra equation (9). The problem (1)-(3) can be solved according to theorem (2.1). For proof its uniqueness, we suppose the reverse, the problem under consideration has two distinct solutions, $\{f_1, g_1\}$ and $\{f_2, g_2\}$. For all points of J, we suggest that $g_1 \neq g_2$. In fact if $g_1 = g_2$ on J, we get $f_1 = f_2$.

The functions g_1 and g_2 produce two distinct solutions to the equation (9), as both pairs achieve identity (6). The uniqueness of equation (9)'s solutions is contradicted by this. The theorem became proved.

Conclusions

With respect to functions and nonlocal conditions, an inverse fractional abstract Cauchy problem in a Hibert space is the main subject of this paper. When the conditions are achieved the existence and uniqueness theorems are Proven. With respect to functions and nonlocal conditions, an inverse fractional abstract Cauchy problem is solved.

Acknowledgments

The chief editor and the referees' anonymous comments and ideas are greatly appreciated by the authors, who would want to convey their appreciation.

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